<u>Common substitutions in integration</u>

	Integral	Substitution /Integrand	dx → dt	Examples
1	$\int R(x,\sqrt{b^2x^2+a^2})\,dx$	$x=\frac{a}{b}\tan t$ $\sqrt{b^2x^2+a^2}=a\sec t$	$dx=\frac{a}{b}\sec^2 t\ dt$	$\int\frac{dx}{x\sqrt{9+4x^2}}=\frac{1}{3}\ln\left\lvert\frac{\sqrt{9+4x^2}-3}{x}\right\rvert+C$ $\int\frac{\sqrt{x^2-25}}{x^4}dx=\frac{(x^2-25)^{3/2}}{75x^3}+C$ $\int\frac{(16-9x^2)^{3/2}}{x^6}dx=-\frac{(16-9x^2)^{5/2}}{80x^5}+C$
2	$\int R(x,\sqrt{b^2x^2-a^2})\,dx$	$x=\frac{a}{b}\sec t$ $\sqrt{b^2x^2-a^2}=a\tan t$	$dx=\frac{a}{b}\sec t\tan t\ dt$	
3	$\int R(x,\sqrt{a^2-b^2x^2})dx$	$x=\frac{a}{b}\sin t$ $\sqrt{a^2-b^2x^2}=a\cos t$	$dx=\frac{a}{b}\cos t\ dt$	
4	$\int\frac{dx}{\sqrt{ax^2+bx+c}}$ $\int\frac{dx}{ax^2+bx+c}$	$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a}$ $=a[t^2\pm\lambda^2], t=x+\frac{b}{2a}, \pm\lambda^2=-\frac{\Delta}{4a^2}$	$dx=dt$ (change to **1-3** above)	$\int\frac{dx}{8x^2+8x+9}=\frac{1}{\sqrt{56}}\tan^{-1}\frac{(2x+1)\sqrt{2}}{\sqrt{7}}+C$ $\int\frac{dx}{\sqrt{20+8x-x^2}}=\sin^{-1}\left(\frac{x-4}{6}\right)+C$ $\int\frac{(7x-9)dx}{x^2-2x+35}=\frac{7}{2}ln\lvert x^2-2x+35\rvert-\frac{2}{\sqrt{34}}tan^{-1}\frac{x-1}{\sqrt{34}}+C$
5	$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ $\int\frac{Ax+B}{ax^2+bx+c}dx$	$Ax+B=\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)$ $\int\frac{Ax+B}{ax^2+bx+c}dx=\frac{A}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{ax^2+bx+c}$		
6	$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$	**Euler's Substituton** $a>0$, $\begin{cases}\sqrt{ax^2+bx+c}=\pm\sqrt{a}x+t\\ x=\frac{t^2-c}{\pm 2\sqrt{a}t+b}\end{cases}$ $c>0$ $\begin{cases}\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}\\ x=-\frac{\pm 2\sqrt{c}t-b}{t^2-a}\end{cases}$	dx $=2\frac{\pm\sqrt{a}t^2+bt\pm c\sqrt{a}}{\left(\pm 2\sqrt{a}t+b\right)^2}dt$ dx $=2\frac{\pm\sqrt{c}t^2-bt\pm a\sqrt{c}}{\left(t^2-a\right)^2}dt$	$\int\frac{dx}{\sqrt{x^2+a}}=\ln\left\lvert x+\sqrt{x^2+a}\right\rvert+C$ $\int\frac{dx}{x^2\sqrt{x^2+2x-1}}=2\tan^{-1}\left(x+\sqrt{x^2+2x-1}\right)$ $+\frac{x-1+\sqrt{x^2+2x-1}}{x+\left[x+1+\sqrt{x^2+2x-1}\right]}+C$ $Let\quad t=x+\sqrt{x^2+2x-1}$

7	$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ $ax^2+bx+c=a(x-\alpha)(x-\beta),\ \Delta>0$	$\sqrt{ax^2+bx+c}=t(x-\alpha)$ $x=\frac{at^2-\alpha\beta}{t^2-\alpha}$	$dx=\frac{2at(\beta-\alpha)}{(t^2-\alpha)}dt$	$\int\frac{dx}{\sqrt{x^2-3x+2}}=\ln\left\|2x-3+2\sqrt{x^2-3x+2}\right\|+C$
8	$\int R\left(x,\sqrt{(x-a)(b-x)}\right)dx$	$x=a\cos^2\theta+b\sin^2\theta$ $\sqrt{(x-a)(b-x)}=(b-a)\sin\theta\cos\theta$	$dx=2(b-a)\sin\theta\cos\theta$	$\int\frac{dx}{\sqrt{(x-a)(b-x)}}=2\sin^{-1}\sqrt{\frac{x-a}{b-a}}+C,\ where\ \ b>a$ $\int_a^b\frac{xdx}{\sqrt{(x-a)(b-x)}}=(a+b)\frac{\pi}{4}$
9	$\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+e}}\right)dx$	$t=\sqrt[n]{\frac{ax+b}{cx+e}}$ $x=\frac{et^n-b}{a-ct^n}$	$dx=\frac{n(ae-bc)t^{n-1}}{\left(a-ct^n\right)^2}dt$	$\int\sqrt{\frac{4+x}{4-x}}dx=8\tan^{-1}\sqrt{\frac{4+x}{4-x}}-\sqrt{16-x^2}+C$ $\int\frac{dx}{(2x+3)\sqrt{4x+5}}=\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{4x+5}{2x+3}}\right)+C$ $\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$
10	$\int R\left(x,\left(\frac{ax+b}{cx+e}\right)^{m/n},\ldots,\left(\frac{ax+b}{cx+e}\right)^{r/s}\right)dx$	$t^L=\frac{ax+b}{cx+e}$, where L = LCM(n, …, s) $x=\frac{et^L-b}{a-ct^L}$	$dx=\frac{L(ae-bc)t^{L-1}}{\left(a-ct^L\right)^2}dt$	
11	$\int R(\cos x,\sin x)dx$	$t=\tan\frac{x}{2}$ $\sin x=\frac{2t}{1+t^2},\cos x=\frac{1-t^2}{1+t^2}$	$dx=\frac{2dt}{1+t^2}$	$\int\frac{dx}{2+\sin x}=\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\tan\frac{x}{2}+1\right)\right]+C$ $\int\frac{dx}{\cot x+\csc x}=\ln\left\|1+\tan^2\frac{x}{2}\right\|+C$
12	$\int R(\cos^2 x,\sin^2 x)dx$	$t=\tan x$ $\sin^2 x=\frac{t^2}{1+t^2},\cos^2 x=\frac{1}{1+t^2}$	$dx=\frac{dt}{1+t^2}$	$\int\frac{dx}{2\sin^2 x+\cos^2 x}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)+C$
13	$\int\frac{dx}{L\sqrt{Q}},\int\frac{dx}{Q\sqrt{L}},\int\frac{dx}{L_1\sqrt{L_2}},\int\frac{dx}{Q_1\sqrt{Q_2}}$ $L\equiv ax+b,\quad Q\equiv px^2+qx+r$	(1) $L=t^{-1}$ (2) $L=t^2$ (3) $L_1=t^{-1}$ *or* $L_2=t^2$ (4) $\sqrt{Q_1/Q_2}=t$	omitted	$\int\frac{dx}{(x+1)\sqrt{3+x-x^2}}$ $=ln\|1+x\|-ln\left\|3x+5+2\sqrt{3+x-x^2}\right\|+C$